DERIVATION AND SOLUTION OF THE HEAT EQUATION IN 1-D

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Abstract

Heat flows in the direction of decreasing temperature, that is, from hot to cool. In this paper we derive the heat equation and consider the flow of heat along a metal rod. The rod allows us to consider the temperature, u(x,t), as one dimensional in x but changing in time, t.

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1 Introduction

The heat equation is an important partial differential equation (PDE) which describes the distribution of heat (or variation in temperature) in a given region over time. Heat is a process of energy transfer as a result of temperature difference between the two points. Thus, the term 'heat' is used to describe the energy transferred through the heating process. Temperature, on the other hand, is a physical property of matter that describes the hotness or coldness of an object or environment. Therefore, no heat would be exchanged between bodies of the same temperature, Christopher Yaluma [6]. In an object, heat will flow in the direction of decreasing temperature. The heat flow is proportional to the temperature gradient, that is;

$$-k\frac{\partial u}{\partial x}$$

where k is a constant of proportionality. Consider a small element of the rod between the positions x and x+δx. The amount of heat in the element, at time t, is

$$H(t)=\sigma\varrho\, u(x,t)\delta x,$$

where σ is the specific heat of the rod and ϱ is the mass per unit length. At time t+δt, the amount of heat is

$$H(t+\delta t)=\sigma\varrho\, u(x,t+\delta t)\delta x$$

Thus, the change in heat is simply

$$H(t+\delta t)-H(t)=\delta\rho(u(x,t+\delta t)-u(x,t))\,\delta x$$

This change of heat must equal the heat flowing in at x minus the heat flowing out at x+δx during the time interval δt. This may be expressed as

$$\left[\left(-k\frac{\partial u}{\partial x}\right)_{x}-\left(-k\frac{\partial u}{\partial x}\right)_{x+\delta x}\right]\delta t$$

Equating these expressions and dividing by δx and δt gives,

$$\delta\rho\frac{u(x,t+\delta t)-u(x,t)}{\delta t}=k\frac{(\partial u/\partial x)_{x+\delta x}-(\partial u/\partial x)}{\delta x}$$

Taking the limits of δx and δt tending to zero, we obtain the partial derivatives, John Fritz et al [3]. Hence, the heat equation in 1-D is

$$\frac{\partial u}{\partial t}=c^2\frac{\partial^2 u}{\partial x^2}$$

where $c^2=k/\sigma\varrho$ is the constant thermal conductivity and $\partial^2u/\partial x^2$ is the thermal conduction. This is in the form, Evans et al [2];

$$u_t = c^2 u_{xx} \tag{1}$$

The heat equation has the same form as the equation describing diffusion, Thambynayagam [4].

By separation of variables;

Let

$$U=XT,\ X=X(x),\ T=T(t) \tag{2}$$

Equation (1) now becomes;

$$XT'=c^2X''T \tag{3}$$

Separating the variables;

$$\frac{1}{c^2}\frac{T'}{T}=\frac{X''}{X}=k \tag{4}$$

Thus;

$$X''-kX=0 \tag{5}$$

$$T'-c^2kT=0 \tag{6}$$

(i) For k=0; $X''=0\Rightarrow X'=C_1\Rightarrow X=C_1x+C_2$ and $T'=0\Rightarrow T=C_3$

$$\Rightarrow u=XT=(C_1x+C_2)C_3=C_1C_3x+C_2C_3 \tag{7}$$

(ii) For k > 0, say $k=\varrho^2$, $X''-\varrho^2X=0$ and $\frac{T'}{T}=c^2\varrho^2$

$$\Rightarrow r=\pm\varrho \Rightarrow X=C_1e^{\varrho x}+C_2e^{-\varrho x}$$

$$\Rightarrow \ln T=c^2\varrho^2t+C_3\Rightarrow T=e^{c^2\varrho^2t+C_3}\Rightarrow T=C_3e^{c^2\varrho^2t}$$

$$\Rightarrow U=XT=(C_1e^{\varrho x}+C_2e^{-\varrho x})C_3e^{c^2\varrho^2t} \tag{8}$$

(iii) For k < 0, say $k=-\varrho^2, X''+\varrho^2 X=0 \Rightarrow r=\pm i\varrho$

$$\Rightarrow \frac{T'}{T}=-c^2\varrho^2 \Rightarrow \ln T=-c^2\varrho^2 t+C_3$$

$$\Rightarrow T=e^{-c^2\varrho^2 t+C_3} \Rightarrow T=C_3 e^{-c^2\varrho^2 t}$$

$$\Rightarrow U=XT=(C_1\cos\varrho x+C_2\sin\varrho x)C_3 e^{-c^2\varrho^2 t}$$

$$\Rightarrow U(x,t)=(A\cos\varrho x+B\sin\varrho x)e^{-c^2\varrho^2 t} \qquad (9)$$

This is consistent with the physical nature of the periodic equation.

2 Heat flow in a metal rod

We consider a metal rod with boundary conditions (BC), Carslaw et al [1]

$$x=0;\ U(0,t)=0;\ x=l;\ U(l,t)=0;\forall t$$

With Initial conditions (IC);

$$t=0; U(x,0)=u_0$$

With

$$x=0; U(0,t)=Ae^{-c^2\varrho^2 t}=0, \Rightarrow A=0$$

$$\Rightarrow U(x,t)=B\sin\varrho x e^{-c^2\varrho^2 t}$$

With

$$x=l, U(l,t)=B\sin\varrho l e^{-c^2\varrho^2 t}=0$$

$$\Rightarrow \sin\varrho l=0 \Rightarrow \varrho l=n\pi \Rightarrow \varrho=\frac{n\pi}{l}$$

Thus

$$U_n(x,t)=B_n\sin\frac{n\pi x}{l}e^{-c^2(\frac{n\pi}{l})^2 t}$$

This can be generalized as

$$U_n(x,t)=b_n \sin \frac{n\pi x}{l} e^{-c^2(\frac{n\pi}{l})^2 t}$$

with

$$b_n=B_n$$

Thus the general solution is

$$U(x,t)=\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} e^{-c^2(\frac{n\pi}{l})^2 t} \tag{10}$$

From the Initial condition, we have

$$U(x,0)=\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}=u_0$$

So that

$$u_0=\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \tag{11}$$

This is just the half range sine series, Weisstein et al [] where;

$$b_n=\frac{2}{l}\int_0^1 f(x) \sin \frac{n\pi x}{l} dx \tag{12}$$

for all positive integers, n

3 Conclusion

It is worth noting that because every term in the solution for U(x,t) has a negativeexponential in it, the temperature must decrease in time and the final solution willtend to U=0. This is different from the wave equation where the oscillations simply continued for all time. This trivial solution, U=0, is a consequence of the particular boundary conditions chosen here.

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